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March 6, 2025

1 Defining Homotopy Groups

Homotopy Let $f, g : X \to Y$ be cts functions. Then a **homotopy** between them is a continuos map

$$H : X \times [0,1] \to Y$$

such that $H(-,0) = f$ and $H(-,1) = g$. This is an equivilence relation, which is moreover compatible with composition, that is

$$X \xrightarrow{f_1, g_1} Y \xrightarrow{f_2, g_2} Z$$

where $f_1 \simeq g_1$ and $f_2 \simeq g_2$ then $f_1 \circ f_1 \simeq g_1 \circ g_2$. Note that a homotopy is then an element of

$$\text{Hom}([0,1] \times X, Y) \cong \text{Hom}([0,1], \text{Hom}(X,Y))$$

that is a homotopy is merely a path in $\text{Hom}(X,Y)$ between the two maps.

If $A \subseteq X$ then we call H a **homotopy rel** (relative) A if $H(-,t)|_A$ is independent of A.

A **retraction** of X onto $A \subseteq X$ is a map $X \to X$ such that $r(X) = A$ and $r|_A = \text{id}$. A deformation retract is a retract that is homotopic to the identity on X. Note that in particular this is a homotopy equvilence of a space onto a subspace.

Category of Triples We now work in the category of triples of spaces, tTop, objects are tripples $x_0 \subseteq A \subseteq X$, where usually x_0 is a point. A map

$$f : (X, A, x_0) \to (Y, B, y_0)$$

is a continuous function such that $f(x_0) \subseteq y_0, f(A) \subseteq B, f(X) \subseteq Y$. Note that an important subcategory of pTop is the category of pointed spaces with pointed maps, where x_0 is a point and A is also the point x_0.

Universal Property of the Quotient If we have $\sim$ an equivilence relation on a space X and $\pi : X \to X/\sim$ is the quotient map then there is a bijection between

$$\left\{ \sim \ \textit{invariant maps } f : X \to Y \right\} \leftrightarrow \left\{ \textit{maps } \bar{f} : X/\sim \to Y \right\}$$

the bijection is simply given by $f \mapsto \bar{f} \circ \pi$. This is summarised in the diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle \pi}\downarrow & \overset{f}{\searrow} & \\ X/\sim & \xrightarrow[\bar{f}]{} & Y \end{array}$$

We can apply this to homotopy, as a map $X \times I \to Y$. If we have a subspace for example $A \subseteq X \times I$ and a map $\big(X \times I\big)/A \to Y$ then by the universal property we get a homotopy, nothing more than a map $X \times I \to Y$. We know that this homotopy has to be invariant under the equivalence relation generated by being in the subspace A. Explicitly we get

$$\begin{array}{ccc} X \times I & & \\ {\scriptstyle \pi}\downarrow & \overset{\exists! f}{\searrow} & \\ X \times I/A & \xrightarrow[\bar{f}]{} & Y \end{array}$$

If we now define 1

$$D^{n+1} \quad := \quad \frac{D^n \times I}{D^n \times \{1\} \cup S^{n-1} \times I}.$$

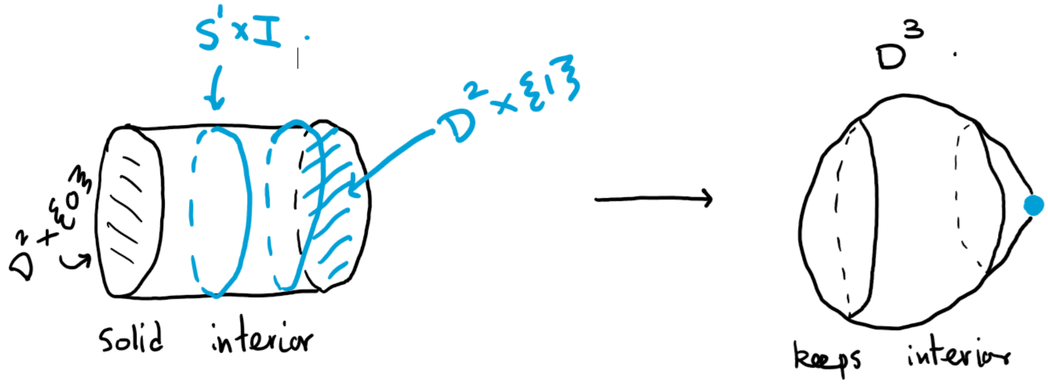


Figure 1: The quotient of the 2 disk cross an interval giving the three disk.

Then we can apply the observation about homotopies to conflate maps on D^{n+1} with homotopies on D^n. That is given a map $D^{n+1} \to X$ we get a unique map $D^n \times I \to X$, but moreover it must be invariant on $D^n \times \{1\} \cup S^{n-1} \times I$. The invariance on $D^n \times \{0\}$ means that the homotopy is between the constant map on D^n and something else, moreover the constant value on $S^{n-1} \times I$, thus it is a homotopy rel the boundary. To summarise we have used the universal property of the quotient to see that there is a bijection between

$$\{\text{maps homotopic to the constant map rel boundary } D^n \to X\} \leftrightarrow \{\text{maps } D^{n+1} \to X\} \leftrightarrow 0 \in \pi_n(X, A)$$

This will be very important in checking the exactness of the LES. The key point is that a map homotopic to the identity rel the boundary is the zero element in $\pi_n(X, A)$.

Relative Homotopy Let (X, A, x_0) be a triple. $p = (1, 0, \cdots, 0) \in S^{n-1} \subseteq D^n$ is also a triple. The **relative homotopy groups** for $n \geq 1$ are then

$$\pi_n(X, A, x_0) \ := \ [D^n, S^{n-1}, p; X, A, x_0]$$

That is *based* homotopy classes of base point preserving maps $(D^n, S^{n-1}) \to (X, A)$. That is if we rephrased our definition of homotopy to be category dependent (i.e. a continuous *morphism* in some subcategory of Top) then these are just homotopy classes of maps. Note that the relative homotopy groups give us the **absolute homotopy groups**

$$\pi_n(X) = \pi_n(X, x_0) \ := \ \pi_n(X, x_0, x_0)$$

This requires D^n and S^{n-1} to map to x_0 and so we can descend to the quotient

$$\pi_n(X) = [D^n/S^{n-1}, s_0; X, x_0] = [S^n, s_0; X, x_0]$$

So the absolute homotopy groups are maps of spheres into the space as we expect.

Relative homotopy defines a functor, suppressing base points from the notation: If $f : (X, A) \to (Y, B)$ then $\pi_n(f) : \pi_n(X, A) \to \pi_n(Y, B)$ is given by

$$[\varphi] \mapsto [f \circ \varphi]$$

Group Structures For absolute homotopy we have the notion $\pi_0(X, x_0)$ by defining it to be $[*, \emptyset, \emptyset; X, x_0, x_0]$ which is homotopy classes of maps $* \to X$ with no conditions. That is just the path components of X.

$\pi_1(X, x_0) = [[0, 1], \{0, 1\}, 0; X, x_0, x_0]$ has a well defined group structure, indeed it is the fundamental group, given by concatenating loops. Consider the quotient of S^n by the equator that is homeomorphic to S^{n-1}, call this quotient map q. Then the quotient is homeomorphic to $S^n \vee S^n$. The sum of two maps can then be written as

$$f + g \ := \ (f \vee g) \circ q$$

Likewise for the relative homotopy groups π_i for $i > 1$ the addition is given by

$$(f + g) \ := \ (f \vee g) \circ q$$

where $c : D^n \to D^n \vee D^n$ is the quotient map given by collapsing the equator $D^{n-1} \subseteq D^n$ to a point1.

In the $i = 1$ case we see that the relative homotopy group is $[[0, 1], \{0, 1\}, 0; X, A, x_0]$, and so is paths that send 0 to x_0. Notice that there is no condition on the location of 1 and so there is no natural way to compose the maps.

Thus we can see that π_n is a functor that lands in *Sets* for $n = 1$, groups for $n = 2$ and Abelian groups for $n > 2$.

2 A LES for Pairs

Consider the inclusion $i : (A, x_0) \to (X, x_0)$ and the inclusion $j : (X, x_0, x_0) \to (X, A, x_0)$. There is also the so called *boundary map*, that is

$$\partial : \text{Hom}((D^n, S^{n-1}, p), (X, A, x_0)) \to \text{Hom}((S^{n-1}, S^{n-1}, p), (X, A, x_0))$$

given by restricting the maps domain. This obviously induces a map up to homotopy and thus a map on $\partial : \pi_n(X, A) \to \pi_{n-1}(A, x_0)$.

Under our identification of homotopies of D^n and maps on D^{n+1} we see that the boundary is

$$\partial(\bar{f} \circ q) = f|_{D^n \times \{0\}}$$

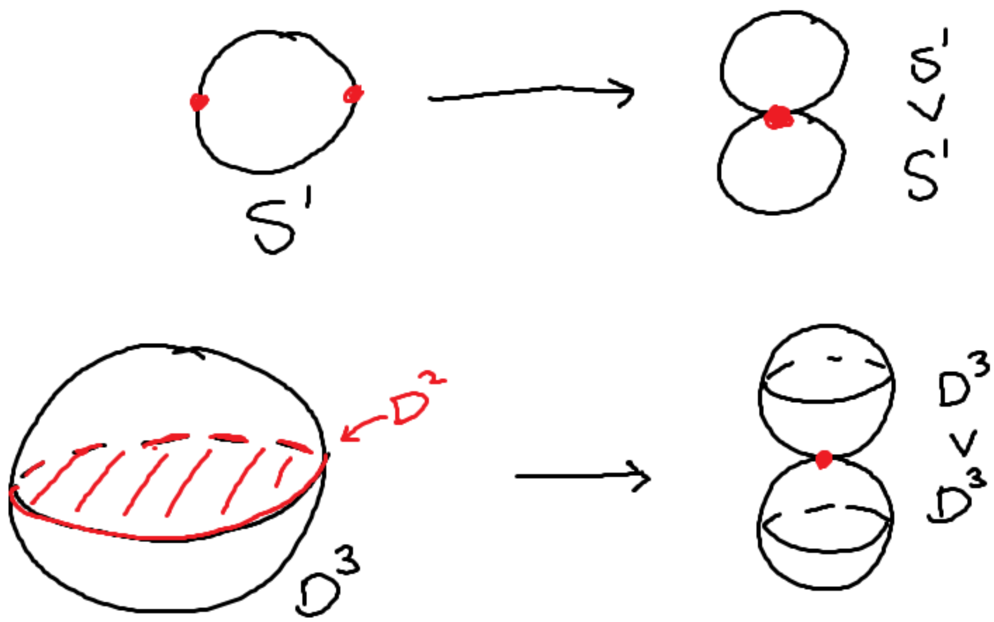


Figure 2: The collapsing maps.

Where $\bar{f}$ is a map on D^{n+1} and f is its lifted homotopy.

Given a pointed set (X, x_0) then we call the kernel of a map into X simply the preimage of the special point x_0. Note that the category of pointed sets is enriched over itself, that is

$$\mathrm{Hom}((X, x_0), (Y, y_0))$$

is a pointed set with the special point being the constant map

$$X \to Y, x \mapsto y_0$$

Theorem. *There is a long exact sequence*

$$\pi_n(A, x_0) \xrightarrow{i_n} \pi_n(X, x_0) \xrightarrow{j_n} \pi_n(X, A, x_0) \xrightarrow{\partial_n}$$
$$\pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}(X, x_0) \xrightarrow{j_*} \pi_{n-1}(X, A, x_0) \dashrightarrow \cdots \dashrightarrow$$
$$\pi_2(A, x_0) \xrightarrow{i_*} \pi_2(X, x_0) \xrightarrow{j_*} \pi_2(X, A, x_0) \xrightarrow{\partial}$$
$$\pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial}$$
$$\pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)$$

The first step in proving this long exact sequence is to understand the kernel, that is which maps are homotopic to the identity. The so called contraction lemma is helpful here:

Lemma. *A map*

$$f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$$

*is zero in $\pi_n(X, A, x_0)$ **iff** f is homotopic to the constant map $D^n \mapsto x_0$ (definition) **iff** f is homotopic relative to S^{n-1} to a map with image contained in A, that is $\exists g \in [f] \in \pi_n(X, A)$ such that $Im(g) \subseteq A$.*

> ***Proof.*** $\Longleftarrow$: Assume such a g exists, then $[f] = [g] \in \pi_n(X, A)$. Note that D^n is contractable and in particular deformation retracts onto its base point s_0. If $r : D^n \to D^n$ is such a retraction, i.e. $r(D^n) = s_0$ (its a retract) such that r is homotopic to the identity on D^n (this is the deformation part); then clearly b/c it is homotopic to the identity we have
>
> $$[g] = [g \circ r]$$
>
> But r is simply the constant map $D^n \mapsto s_0$ and $g(s_0) = x_0$ and so $g \circ r : D^n \mapsto x_0$ and is therefore zero in $\pi_n(X, A)$.
>
> $\Longrightarrow$: Notice that even though $f \simeq 0$, unless $f = 0$, this homotopy is not *rel* S^{n-1}. Thus this direction is not immediate.
>
> The key is the following homeomorphism: Consider $G_t : D^n \to D^n \times \{t\} \cup S^{n-1} \times [0, 1] \subseteq D^n \times [0, 1]$ given by *stretching the disk over the cylinder*. The key properties are that $\partial D^n = S^{n-1} \mapsto S^{n-1} \times \{0\}$ and that it is a homeomorphism. This defines a map $G : D^n \times [0, 1] \to D^n \times [0, 1]$ that is a homeomorphism for each $t \in [0, 1]$. 2
>
> Now let $F : D^n \times [0, 1] \to X$ be the homotopy between f and 0. Then we claim that $F \circ G : D^n \times [0, 1] \to X$ is a homotopy rel S^{n-1} from f to a map with image in A.
>
> First we check the image of the map that we homotope to, i.e. $F \circ G(-, 1)$. This map is just $F|_{D^n \times \{1\} \cup S^{n-1} \times [0,1]}$. So look at the two peices of the domain, $F(-, 1) = 0$ and so has image x_0 in particular in A. Again $F(s, t)$ for $s \in S^{n-1}$ gives the value of a map of tripples $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ and so maps values in S^{n-1} to values in A.
>
> Next check the it fixes pointwise S^{n-1}: Because $S^n - 1$ is sent to $S^{n-1} \times \{0\}$ by $G(-, t)$ for every t we see that $F \circ G$ is independent of t.

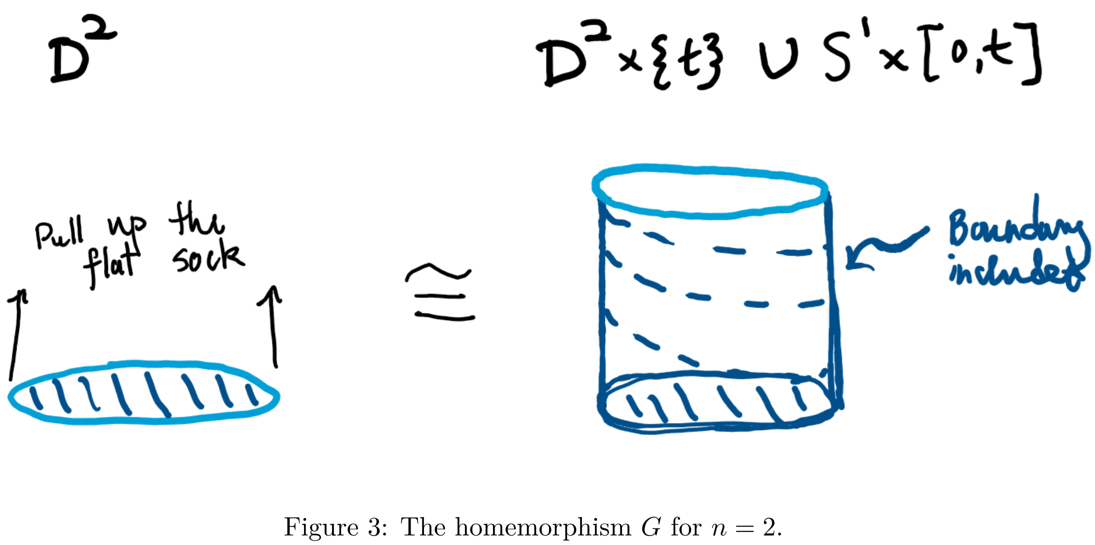


Figure 3: The homemorphism G for $n = 2$.

Clearly this lemma is saying something like if f is equivalent to something coming from $\pi_n(A, x_0)$ then it must be zero. We now need to check:

$$\ker(i_n) = \text{Im}(\partial_{n+1}), \quad n \geq 0$$

Figure 4: The map f being homotoped rel the boundary to g.

$$\ker(\partial_n) = \mathrm{Im}(j_n), \quad n \geq 1$$
$$\ker(j_n) = \mathrm{Im}(i_n), \quad n \geq 1$$

which we will do by showing that ker $\subseteq$ Im and Im $\subseteq$ ker for each case.

i_* **and** ∂_***:** $\ker(i_n) \subseteq \mathrm{Im}(\partial_{n+1})$: Let $f \in \pi_n(A, x_0)$ such that $i_* f \simeq 0 \in \pi_n(X, x_0)$. Then take $F : D^n \times I \to X$ to be the homotopy from f to 0. Lets check that F sends $D^n \times \{1\} \cup S^{n-1} \times I$ to x_0. First $F(-, 0) = f \in \pi_n(A, x_0)$ so sends the boundary S^{n-1} to the basepoint, moreover it is a homotopy to the zero map so indeed. Therefore by the universal property of the quotient

can I conclude that the homotopy doesnt chaange the value along the boundary from x0 and then go back?

$$q \circ F \in \pi_{n+1}(X, A, x_0)$$

and moreover

$$\partial(q \circ F) = F|_{D^n \times \{0\}} = f$$

$\ker(i_n) \supseteq \mathrm{Im}(\partial_{n+1})$: We consider $f \in \pi_{n+1}(X, A, x_0)$ and compose

$$i_* \partial_*(f) = f|_{D^n \times \{0\}}$$

Now we need a homotopy from $f|_{D^n \times \{0\}}$ to 0. This exists by the universal property of the quotient.

j_* **and** ∂_***:** $\ker(\partial_n) \subseteq \mathrm{Im}(j_n)$: *The statement is essentially that; If the boundary of f is homotopic to the identity then f is homotopic to a map whose boundary is zero.* Let $f \in \pi_n(X, A, x_0)$ such that $\partial f = [\partial f] \simeq 0 \in \pi_{n-1}(A, x_0)$. Take $F : D^{n-1} \times I \to X$ to be the homotopy from $0 \to \partial f$, We apply the universal property of the quotient to get the lift of f, $\hat{f} : D^{n-1} \times I \to X$. We then compose these homotopies

$$\hat{f} + F(x,t) \quad := \quad \begin{cases} F(x, 2t), & t \in \frac{1}{2}I \\ \hat{f}(x, 2(t - \frac{1}{2})), & t \in [\frac{1}{2}, 1] \end{cases}$$

This then corresponds again to a map on the quotient $\hat{f} + F : D^n \to X$. It is clear that $\partial(\hat{f} + F) = F(x, 0) = 0$. By construction we have attached a null homotopic map to f and so the result is still homotopic to f.

$\ker(\partial_n) \supseteq \mathrm{Im}(j_n)$: Take $f \in \pi_n(X, x_0)$ and apply both maps

$$\partial_* j_* f = \partial f$$

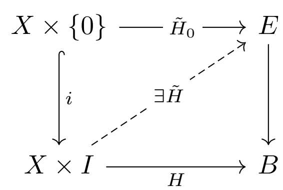
But f is from the absolute homotopy group on X which means all of S^{n-1} goes to zero, hence $\partial f = 0$.

i_* **and** j_***:** This is essentially the compression lemma. $\ker(j_n) \subseteq \text{Im}(i_n)$ becuase if a map is in the kernel of j_* then by definition it is zero in $\pi_n(X, A, x_0)$. Therefore by the lemma it is homotopic rel the boundary to an element of $\pi_n(A, x_0)$, that is a map whose image is contained in A. Hence it is in the image of i_*.

For $\text{Im}(i_n) \subseteq \ker(j_n)$, we see that being in the image of i_* means that there is a map in $\pi_n(A, x_0)$ to which we apply i. Hence the map is homotopic to one whose image is contained in A. Therefore by the compression lemma it is zero in $\pi(X, A, x_0)$.

3 A LES for Fibrations

Given a bundle (a surjection) $E \to B$ we say that it has the homotopy lifting property for a space X if



that is *for every homotopy into the base if there is a lift of one of the maps then there is a lift of the homotopy.*

A bundle is a Serre fibration if it has the homotopy lifting property for all X that are *CW complexes*, or equivalently for disks. If a space has the homotopy lifting property for all spaces then it is called a *Hurewicz* fibration. The homotopy lifting property ensures that the fibers are all homotopic. We will work in the more general setting of a Serre fibration (b/c all spaces are weakly homotopic to CW complexes this is largely sufficient for homotopy theory). Consider such a fibration

$$F \hookrightarrow E \xrightarrow{p} B$$

Applying the LES of homotopy groups to the pair (E, F) gives

$$\pi_n(F) \to \pi_n(E) \to \pi_n(E, F)$$

There is also a natural map of pairs $(E, F) \to (B, *)$ given by p, which enduces a map of homotopy groups (by functoriality)

$$\pi_n(E, F) \to \pi_n(B)$$
$$f \mapsto p \circ f$$

We claim that this map is an isomorphism.

The key insight is that the homotopy lifting property is stronger than it first appears. Homotopy lifting says that fixing a lift of one end of a homotopy will allow you to lift the homotopy but you do not have control of what the other end will lift to. Diagramatically 3 We can only control the lift of the first edge, the other lifts are only in the fiber of where we want them to be. That is they project to what we started with but might be anywhere in the preimage.

If we define the homotopy lifting property of a pair (X, A) to be that given a homotopy $H : X \times I \to B$ and a lift of both $H(-, 0)$ and $H|_A$ then there is a lift of all of H extending these two conditions, then

Lemma (Hatcher, Thm 4.41)**.** *Homotopy lifting for D^n is equivilent to homotopy lifting for $(D^n, \partial D^n)$*

Proof. The pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup \partial D^n \times I)$ are homemorphic 2. This homemorphism means that a homotopy, exactly a map on $D^n \times I$, if we have fixed the first edge, exactly a lift of the homotopy at $D^n \times \{0\}$ can equally be considered as a map from $D^n \times I$ that has fixed the first edge and the boundary.

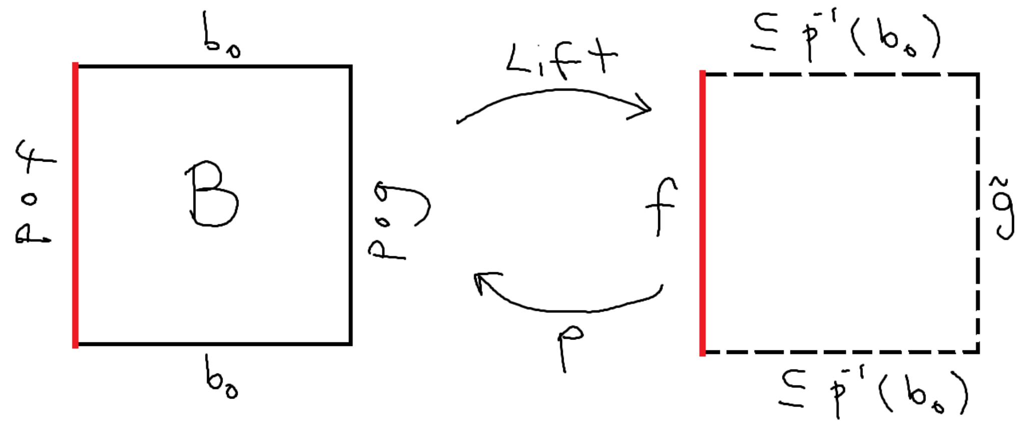
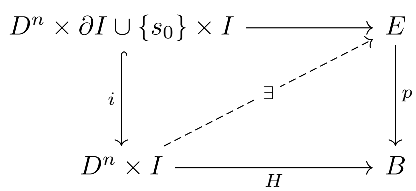


Figure 5: Only control a priori the lift of one of the edges.

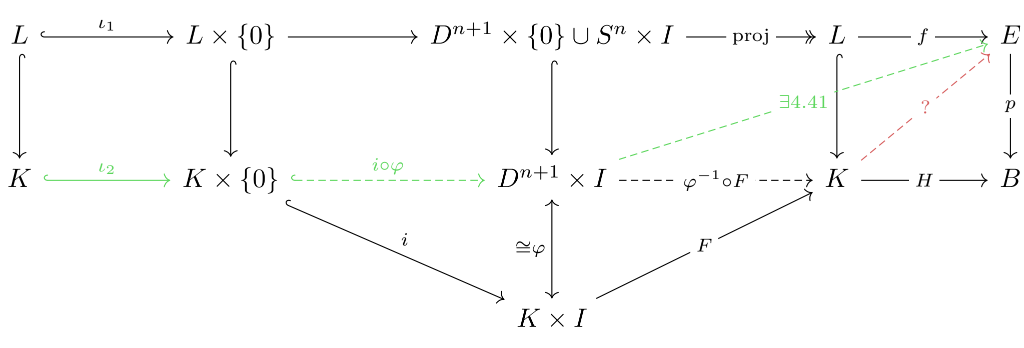
In essence this lemma tells us that homotopy lifting allows us to fix not just the lift of $p \circ f$ but also of b_0 in the diagram of 3. This can now be used to give us control over both ends!

Lemma (Bredon, Thm 6.4)**.** *Let $E \to B$ be a fibration. The following commutive diagram can be completed 3*



Note that the map on the top edge is a specified lift of part of F.

Proof: Let $K = D^n \times I$ and $L = D^n \times \partial I \cup \{s_0\} \times I$ and stare at the following



The idea is that we can lift a homotopy from D^{n+1} and so we will include K, a homotopy from D^n, into the first face of this. Then when we lift the homotopy on this face we are having control over the whole lift of K. The goal is to give the red arrow, that extends f and projects to H.

Formally: Let $F : K \times I \to K$ be the strong deformation retract from $K \times I$ onto $D^{n+1} \times \{0\} \cup S^n \times I$. In particular F has the following properties:

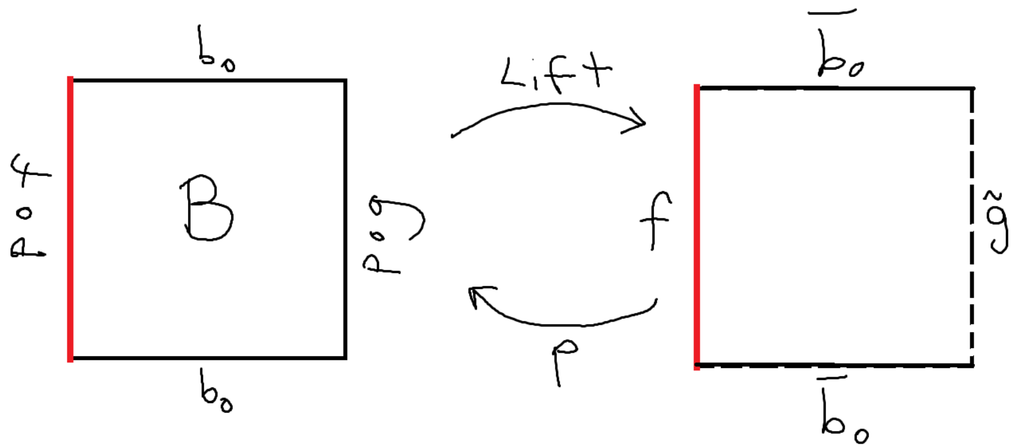


Figure 6: Only control a priori the lift of one of the edges.

- $F(-,0) = \mathrm{id}_K$ (retract)
- $F(-,1) \in L$ (onto L)
- $\forall l \in L, \forall t \in I$ we have $F(l,t) \in L$ (strong)

By our previous theorem we have a lift of the map $H \circ \varphi^{-1} \circ F$, which we call ψ. Now we claim that $\psi \circ i \circ \varphi \circ \iota_2 : K \to E$ (the green) is the desired lift. We need to check that it restricts on L to f and that when projecting we get H.

$$\begin{aligned} p \circ \psi \circ i \circ \varphi \circ \iota_2(k) &= p \circ \psi \circ i \circ \varphi(k,0) \\ \text{(commutativity from Thm 4.41)} \quad &= HFi(k,0) \\ \text{(F is retract)} \quad &= H(k) \end{aligned}$$

and for $l \in L$

$$\begin{aligned} \psi \circ i \circ \varphi \circ \iota_2(l) &= \psi \circ i \circ \varphi(l,0) \\ \text{(commutativity from Thm 4.41)} \quad &= f \circ \mathrm{proj} \circ \iota_3(l) \\ \text{(proj is left inverse of } \iota_3) \quad &= f(l) \end{aligned}$$

□

Now we can prove the isomorphism we claimed.

Injection: Assume $[pf] = [pg]$. Then there is a homotopy between them

$$H : D^n \times I \to B$$

which we specify the lift at $D^n \times \{0\}$ to be f and the lift of $D^n \times \{1\}$ to be g. By the previous theorem we can lift the whole map and therefore f and g are homotopic.

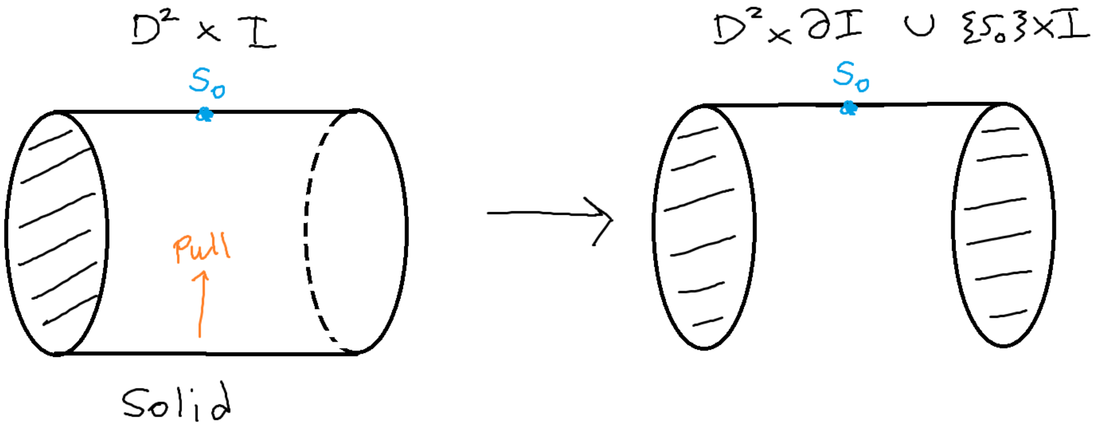


Figure 7: Deformation retract of a cylinder onto its ends and an edge.

Surjection: Consider an element $f \in \pi_n(B)$. This is a map $(D^n, S^{n-1}, s_0) \to (B, b_0, b_0)$. Apply the universal property of the quotient to get the coresponding homotopy, to the trivial map. We can lift the constant map to the constant map into E. Now by homotopy lifting we get a lift of f (considered as a homotopy), call it $\tilde{f}$. Then we need to check that this lift is an element of $\pi_n(E, F)$, however since $p\tilde{f}(\partial D^n) = b_0$ we know that the lift is in the fiber over b_0, hence by definition in F. So we are done.

this might not be sufficient in lower degrees (n=1)

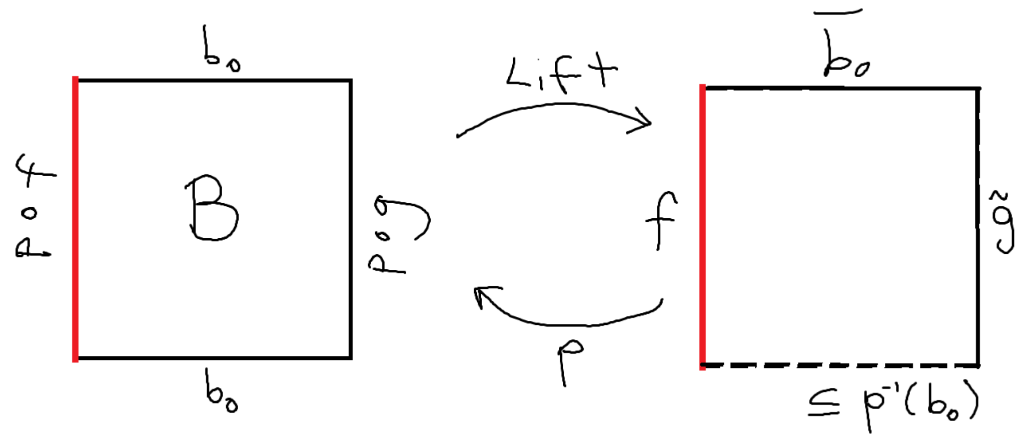


Figure 8: The strong homotopy lifting property.

References